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IC/98/140

United Nations Educational Scientific and Cultural Organization  
and  
International Atomic Energy Agency  
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

## ON THE QUANTUM SUPER VIRASORO ALGEBRA

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### Abstract

The quantum super-algebra structure on the deformed super Virasoro algebra is investigated. More specifically we established the possibility of defining a non trivial Hopf super-algebra on both one and two-parameters deformed super Virasoro algebras.

MIRAMARE – TRIESTE

September 1998

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Quantum algebras (quantum groups)[1, 2], are generalizations of the usual Lie algebras to which they reduce for appropriate values of the deformation parameters. From the mathematical point of view they are Hopf algebras [3]. Their use in physics became popular with the introduction of the deformed harmonic oscillator as a tool for providing a realization of the quantum (super) algebra [4-6]. Particularly, a great deal of attention has been paid to the deformation of the Virasoro algebra [7-14] and the super Virasoro algebra [15-18]. Several investigations of the latter super-algebra are based on the coordinates and derivatives on super space [15-16]. However, the existence of a quantum super-algebra structure of the deformed super Virasoro algebra is still an open question. The purpose of this paper is to give a realization of the deformed super Virasoro algebra in terms of bosonic and fermionic algebra operators and define a quantum super-algebra structure on this deformed super-algebra. In ref. [19], an algebraic deformation (associated to an R-matrix of unit square) which allows to a non trivial Hopf super-algebra structure on the super Virasoro and Ramond-Neveu-Schwarz algebras will be investigated.

First, let us recall that the classical super Virasoro algebra ( $sVir$ ) is defined as an infinite super-algebra generated by the generators  $\{L_n, G_n, F_n; n \in \mathbb{Z}\}$  satisfying the defining relations

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m}, \\ [F_m, G_n] &= G_{n+m}, \\ [L_n, F_m] &= -mF_{n+m}, \\ [F_m, F_n] &= 0, \\ [L_m, G_n] &= (m - n)G_{n+m}, \\ [G_m, G_n] &= 0, \end{aligned} \tag{1}$$

and that the Hopf super-algebra structure on the enveloping super-algebra of this classical super-algebra  $U(sVir)$  is trivial. The  $\mathbb{Z}_2$ -grading on this super-algebra is defined by requiring that  $\deg(L_i) = \deg(F_i) = 0$  and  $\deg(G_i) = 1$  for  $(i \in \mathbb{Z})$ . Further, the bracket  $[\cdot, \cdot]$  in (1) stands for a graded one

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)}yx.$$

A possible realization of the classical super Virasoro algebra (1) is given as follows

$$\begin{aligned} L_k &= -(a_+)^{k+1}a_-, \\ G_n &= (a_+)^{n+1}f_+a_-, \\ F_n &= (a_+)^nf_+f_- \end{aligned} \tag{2}$$

where the operators  $\{a_+, a_-, 1\}$  generate the classical bosonic algebra

$$\begin{aligned} a_-a_+ - a_+a_- &= 1 \\ [1, a_{\pm}] &= 0. \end{aligned} \tag{3}$$

and the remaining ones  $\{f_+, f_-, 1\}$  generate the classical fermionic algebra given by

$$\begin{aligned} f_- f_+ + f_+ f_- &= 1 \\ [1, f_\pm] &= 0, \quad (f_\pm)^2 = 0. \end{aligned} \quad (4)$$

A  $q$ -deformed version of the super Virasoro algebra will be defined by the generators  $\{L_n, G_n, K_n; n \in \mathbb{Z}\}$  and the following  $q$ -deformed relations

$$\begin{aligned} q^{l-k} L_l L_k - q^{k-l} L_k L_l &= [l-k] L_{k+l} \\ F_m G_n - G_n F_m &= G_{n+m}, \\ L_l F_k - q^{2k} F_k L_l &= -q[[k]] F_{k+l}, \\ q^{n-m} F_m F_n - q^{m-n} F_n F_m &= \lambda[n-m] F_{n+m}, \\ q^{l-k} L_l G_k - q^{k-l} G_k L_l &= [l-k] G_{k+l}, \\ G_m G_n + G_n G_m &= 0 \end{aligned} \quad (5)$$

(where  $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$ ,  $[[x]] = \frac{1 - q^{2x}}{1 - q^2}$  and  $\lambda = q - q^{-1}$ ).

The  $q$ -deformed super-algebra (5) can be realized in terms of a  $q$ -deformed bosonic algebra generators  $\{a_+, a_-, 1\}$  and a classical fermionic algebra generators  $\{f_+, f_-, 1\}$  which are such that

$$\begin{aligned} [a_-, a_+]_{q^2} &= 1, \quad \{f_-, f_+\} = 1 \\ [a_\pm, f_\pm] &= 0, \quad [a_\mp, f_\pm] = 0, \\ (f_-)^2 &= 0, \quad (f_+)^2 = 0, \\ [1, a_\pm] &= 0, \quad [1, f_\pm] = 0 \end{aligned} \quad (6)$$

(where  $[a, b]_\alpha = ab - \alpha ba$ ).

In fact, the following operators

$$\begin{aligned} L_k &= -q(a_+)^{k+1} a_-, \\ G_n &= (a_+)^{n+1} f_+ a_-, \\ F_n &= (a_+)^n f_+ f_- \end{aligned} \quad (7)$$

generate the  $q$ -deformed super Virasoro algebra (5).

Now we define the quantum super-algebra structure on the  $q$ -deformed enveloping super-algebra of the  $q$ -deformed super Virasoro algebra  $U_q(sVir_q)$  as follows:

$$\begin{aligned} \Delta(L_i) &= L_i \otimes T_i + 1 \otimes L_i, \\ \Delta(G_i) &= G_i \otimes R_i + 1 \otimes G_i, \\ \Delta(F_i) &= F_i \otimes K_i + 1 \otimes F_i, \\ \epsilon(L_i) &= 0, \\ \epsilon(G_i) &= 0, \\ \epsilon(F_i) &= 0, \\ S(L_i) &= -L_i S(T_i), \\ S(G_i) &= -G_i S(R_i), \\ S(F_i) &= -F_i S(K_i), \end{aligned} \quad (8)$$

where the operators  $\{K_i, T_i, R_i, \quad i \in \mathbb{Z}\}$  are additional invertible even elements of  $U_q(sVir_q)$  which reduce to the unity operator when  $q = 1$ .

From the consistence of the costructures on  $q$ -deformed super Virasoro algebra generators with the basic axioms of the Hopf super-algebra structure ( co-associativity, co-unity and antipode ), we deduce that the costructures on the additional elements are given by

$$\begin{aligned}
\Delta(T_i) &= T_i \otimes T_i, \\
\Delta(R_i) &= R_i \otimes R_i, \\
\Delta(K_i) &= K_i \otimes K_i, \\
\epsilon(T_i) &= 1, \\
\epsilon(R_i) &= 1, \\
\epsilon(K_i) &= 1, \\
S(T_i) &= T_i^{-1}, \\
S(R_i) &= R_i^{-1}, \\
S(K_i) &= K_i^{-1}.
\end{aligned} \tag{9}$$

Further, the consistence of the Hopf super-algebra structure (8) with the deformed commutations relations (5), (i.e. using the fact that the coproduct ( $\Delta$ ) and counit ( $\epsilon$ ) operations must be super-algebra homomorphisms, and that the coinverse ( $S$ ) must be super-algebra anti-homomorphism) we obtain the following relations between the  $q$ -deformed super Virasoro algebra generators and the additional elements.

$$\begin{aligned}
L_k T_l &= q^{2(l-k)} T_l L_k, \\
L_k K_l &= q^{2l} K_l L_k, \\
G_k K_l &= K_l G_k, \\
F_k T_l &= q^{-2k} T_l F_k, \\
F_k R_l &= R_l F_k, \\
G_k T_l &= q^{2(l-k)} T_l G_k, \\
L_l R_k &= q^{2(k-l)} R_k L_l, \\
F_k K_l &= q^{2(k-l)} K_l F_k, \\
G_k R_l &= R_l G_k,
\end{aligned} \tag{10}$$

where the additional generators must satisfy the following relations between them

$$\begin{aligned}
T_k T_l &= T_l T_k = T_{k+l}, \\
T_k K_l &= K_l T_k = K_{l+k}, \\
R_k K_l &= K_l R_k, \\
K_k K_l &= K_l K_k = K_{l+k}, \\
R_k R_l &= R_l R_k, \\
T_l R_k &= R_k T_l = R_{k+l}.
\end{aligned} \tag{11}$$

Considering now the two-parameters  $(p, q)$ -deformation version of the super Virasoro algebra given by

$$\begin{aligned}
q^{l-k} L_l L_k - q^{k-l} L_k L_l &= [l-k] L_{k+l}, \\
F_m G_n - G_n F_m &= G_{n+m}, \\
L_l F_k - q^{2k} F_k L_l &= -q[[k]] F_{k+l}, \\
q^{n-m} F_m F_n - q^{m-n} F_n F_m &= \lambda[n-m] F_{n+m}, \\
q^{l-k} L_l G_k - p^{2l} q^{k-l} G_k L_l &= [l-k] G_{k+l}, \\
G_m G_n + G_n G_m &= 0
\end{aligned} \tag{12}$$

which can be realized as

$$\begin{aligned} L_k &= -q(a_+)^{k+1}a_-, \\ G_n &= p^{-2}(a_+)^{n+1}f_+a_-, \\ F_n &= (a_+)^nf_+f_-, \end{aligned} \tag{13}$$

where the generators  $\{a_+, a_-, f_+, f_-, 1\}$  satisfy now

$$\begin{aligned} [a_-, a_+]_{q^2} &= 1, & \{f_-, f_+\} &= 1 \\ [a_+, f_+]_{p^2} &= 0, & [a_-, f_+]_{p^{-2}} &= 0, \\ [a_-, f_-]_{p^2} &= 0, & [a_+, f_-]_{p^{-2}} &= 0, \\ (f_-)^2 &= 0, & (f_+)^2 &= 0. \\ [1, a_\pm] &= 0, & [1, f_\pm] &= 0. \end{aligned} \tag{14}$$

In this case, the  $(p, q)$ -quantum super-algebra structure on the  $U_{p,q}(sVir_{p,q})$ , will be given by the same costructures on the  $q$ -deformed super Virasoro algebra generators (8) and with the same costructures on the additional even generators. But the deformed relations between the  $(p, q)$ -deformed super Virasoro algebra generators and the additional elements are

$$\begin{aligned} L_k T_l &= q^{2(l-k)} T_l L_k, \\ L_k K_l &= q^{2l} K_l L_k, \\ G_k K_l &= K_l G_k, \\ F_k T_l &= q^{-2k} T_l F_k, \\ F_k R_l &= R_l F_k, \\ G_k T_l &= p^{-2l} q^{2(l-k)} T_l G_k, \\ L_l R_k &= p^{2l} q^{2(k-l)} R_k L_l, \\ F_k K_l &= q^{2(k-l)} K_l F_k, \\ G_k R_l &= R_l G_k \end{aligned} \tag{15}$$

where the additional generators  $\{K_i, T_i, R_i, \quad i \in \mathbb{Z}\}$  must satisfy the relations (11) as in the one parameter deformation case.

In concluding we report that we established that it is possible to define a non trivial Hopf super-algebra structure on the deformed super Virasoro algebra, if we extend its deformed enveloping super-algebra with a set of additional even invertible elements.

## Acknowledgments

The author would like to thank Prof. A. El. Hassouni and Prof. F. Guedira for their guidance and encouragement and Prof. R. Floreanini for reading this paper and for useful discussions. He also thanks the Abdus Salam International Centre for Theoretical Physics, Trieste, for hospitality.

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